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Strong Law of Large Numbers in D-Posets

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The notion of an observable and a state on a D-poset have been introduced. In the present paper the independence of a sequence of observables is defined and the strong law of large numbers is proved.

1. INTRODUCTION

D-posets have been introduced as a natural generalization of various models occurring in quantum structures, especially quantum logics and fuzzy quantum logics. A typical example of a D-poset is the set F of all functions $f: X \to \langle 0, 1\rangle$ considered with the partial binary operation \ defined for every pair (f, g) with $f \leq g$ by the formula $g\backslash f(t) = g(t) - f(t)$. The general definition is the following.

Definition 1. Let $(F, \leq)$ be a partially ordered set with the smallest element 0_F and the greatest element 1_F. It is called a D-poset if a partial binary operation \ on F is given such that $b\backslash a$ is defined if and only if $a \leq b$ and the following conditions are satisfied:

(i) If $a \leq b$, then $b\backslash a \leq b$ and $b\backslash b(\backslash a) = a$.
(ii) If $a \leq b \leq c$, $c\backslash b \leq c\backslash a$ and $(c\backslash a)\backslash(c\backslash b) = b\backslash a$.

The notion of D-posets was introduced by Chovanec and Kôpka (1992; Kôpka and Chovanec, 1994) and independently (in another form) by Giuntini and Greuling (1989). It is interesting that simultaneously the very close notion of an orthoalgebra was introduced by Foulis *et al.* (1992). For the relation between D-posets and orthoalgebras see Navara and Pták (n.d.).

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Definition 2. A state m on a D-poset F is a mapping $m: F \to \langle 0, 1\rangle$ satisfying the following conditions:

(i) $m(1_F) = 1$.
(ii) If $f, g \in F, f \leq g$, then $m(g) = m(f) + m(g \backslash f)$.
(iii) If $f_n \in F$ $(n = 1, 2, \ldots), f \in F$, and $f_n \nearrow f$, then $m(f_n) \nearrow m(f)$.

Definition 3. An observable on a D-poset F is a mapping $x: \mathcal{B}(R) \to F$ [where $\mathcal{B}(R)$ is the σ-algebra of Borel subsets of the set R of real numbers] satisfying the following conditions:

(i) $x(R) = 1_F$.
(ii) If $A, B \in \mathcal{B}(R)$, $A \subset B$, then $x(B \backslash A) = x(B) \backslash x(A)$.
(iii) If $A_n \in \mathcal{B}(R)$ $(n = 1, 2, \ldots)$ $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

It is easy to see that the function $m_x = m \circ x: \mathcal{B}(R) \to \langle 0, 1\rangle$ is a probability measure—the probability distribution of the observable x with respect to the state m. Therefore it is natural to define the mean value

$$m(x) = \int_{-\infty}^{\infty} t \, dm_x(t)$$

and the dispersion

$$\sigma^2(x) = \int_{-\infty}^{\infty} t^2 \, dm_x(t) - m(x)^2$$

of course, if the mentioned integrals exist.

The preceding considerations make it possible to formulate and prove some versions of the weak law of large numbers (Chovanec and Jurečková, 1992; Riečan, n.d.-c). Of course, the problem of the strong law must be considered together with the almost everywhere convergence. This was done first in Riečan (n.d.-b) and in a more convenient form (for our purposes) in Riečan (n.d.-d). Mainly we shall use the results of Riečan (n.d.-d) for independent sequences of observables in D-posets.

2. INDEPENDENCE

Definition 4. Let $(x_n)_n$ be a sequence of observables on a D-poset F. We shall say that $(x_n)_n$ is strongly independent if to every $n \in N$ there exists a mapping $h_n: \mathcal{B}(R^n) \to F$ satisfying the following conditions:

(i) $h_n(R^n) = 1_F$.
(ii) If $A, B \in \mathcal{B}(R^n)$ and $A \subset B$, then $h_n(B) \backslash h_n(A) = h_n(B \backslash A)$.
(iii) If $A_i \in \mathcal{B}(R^n)$ $(i = 1, 2, \ldots)$ and $A_i \nearrow A$, then $h_n(A_i) \nearrow h_n(A)$.

(iv) $m(h_n(A_1 \times \cdots \times A_n)) = m(x_1(A_1)) \cdot \ldots \cdot m(x_n(A_n))$ for every $A_1, \ldots, A_n \in \mathcal{B}(R)$.

(v) For every $A, B \in \mathcal{B}(R^n)$ there exists the greatest lower bound $h_n(A) \wedge h_n(B)$.

Let us mention two examples of independent sequences of observables in a D-poset.

Example 1. Let Ω be an arbitrary nonempty set and $\mathcal{F} = \{f: \Omega \to \langle 0, 1\rangle; f \text{ constant}\}$. Define $m: \mathcal{F} \to \langle 0, 1\rangle$, $m(c) = c$. Further, let $(\mu_n)_n$ be a sequence of probability measures on $\mathcal{B}(R)$. For every $A \in \mathcal{B}(R)$ and $\omega \in \Omega$ define $x_n(A)(\omega) = \mu_n(A)$. Evidently $x_n: \mathcal{B}(R) \to F$ is an observable for every $n \in N$. We assert that $(x_n)_n$ is an independent sequence of observables.

Namely we can define $h_n: \mathcal{B}(R^n) \to F$ by the formula $h_n(A)(\omega) = \mu_1 \times \cdots \times \mu_n(A)$. Then

$$
\begin{aligned}
&m(h_n(A_1 \times \cdots \times A_n)) \\
&\quad = \mu_1 \times \cdots \times \mu_n(A_1 \times \cdots \times A_n) \\
&\quad = \mu_1(A_1) \cdot \ldots \cdot \mu_n(A_n) = m(x_1(A_1)) \cdot \ldots \cdot m(x_n(A_n))
\end{aligned}
$$

Example 2. Let F be a D-poset, $y: \mathcal{B}(R) \to F$ be a fixed observable. Let $m_y: \mathcal{B}(R) \to \langle 0, 1\rangle$ be the probability measure defined by the formula $m_y(A) = m(y(A))$. Let $(f_n)_n$ be a sequence of Borel measurable functions, $f_n: R \to R$ independent with respect to m_y, i.e.,

$$
m_n(f_1^{-1}(A_1) \cap \cdots \cap f_n^{-1}(A_n)) = m_y(f_1^{-1}(A)) \cdot \ldots \cdot m_y(f_n^{-1}(A_n))
$$

for every $n \in N$ and every $A_1, \ldots, A_n \in \mathcal{B}(R)$. Define $x_n: \mathcal{B}(R) \to F$ by the formula $x_n(A) = y(f_n^{-1}(A))$. Then $(x_n)_n$ is an independent sequence of observables.

Indeed, if we put $T: R \to R^n$, $T(u) = (f_1(u), \ldots, f_n(u))$ and $h_n = y \circ T^{-1}$, then

$$
\begin{aligned}
&m(h_n(A_1 \times \cdots \times A_n)) \\
&\quad = m_y(f_1^{-1}(A_1) \cap \cdots \cap f_n^{-1}(A_n)) \\
&\quad = m_y(f_1^{-1}(A_1)) \cdot \ldots \cdot m_y(f_n^{-1}(A_n)) = m(x_1(A_1)) \cdot \ldots \cdot m(x_n(A_n))
\end{aligned}
$$

Definition 4 makes it possible to define the arithmetic mean which occurs in the law of large numbers and, of course, more general operations. Put

$$
g_n: \quad R^n \to R, \quad g_n(v_1, \ldots, v_n) = \frac{1}{n} \sum_{i=1}^{n} v_i
$$

Then we define

$$\frac{1}{n}\sum_{i=-1}^{n} x_i = h_n \circ g_n^{-1}$$

This definition is in harmony with the classical case of random variables $\xi_1, \ldots, \xi_n$ and the corresponding random vector $T_n = (\xi_1, \ldots, \xi_n)$. Then

$$\frac{1}{n}(\xi_1 + \cdots + \xi_n) = g_n \circ T_n$$

hence

$$(g_n \circ T_n)^{-1} = T_n^{-1} \circ g_n^{-1}$$

Here an observable x_i: $\mathcal{B}(R) \to F$ substitutes the role of a random variable ξ_i: $\Omega \to R$ [considering the mapping $E \mapsto \xi_i^{-1}(E)$] and h_n: $\mathcal{B}(R^n) \to F$ the role of the random vector $T_n = (\xi_1, \ldots, \xi_n)$ [considering the mapping $A \mapsto T_n^{-1}(A)$].

3. CONVERGENCE

Definition 5. We shall say that a sequence $(y_n)_n$ of observables on F converges to 0 m-almost everywhere if the greatest lower bound $\bigwedge_{n=k}^{k+i} y_n((-1/p, 1/p))$ exists for every $k, i, p \in N$ and

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 1$$

Theorem. Let $(x_n)_n$ be a strongly independent sequence of observables on a D-poset F such that $\sigma^2(x_n)$ exists for every n and $\Sigma_{n=1}^{n} [\sigma^2(x_n)/n^2] < \infty$. Then

$$\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - m(x_i))\right)_{n=1}^{\infty}$$

converges m-a.e. to 0.

Proof. By Riečan (n.d.-d, Section 3) there exists a probability space $(R^N, \mathcal{S}, P)$ such that

$$P(\Pi_n^{-1}(A)) = m(h_n)(A)) \qquad (*)$$

for every $A \in \mathcal{B}(R^n)$ (Π_n: $R^N \to R^n$ is the projection).

Define ξ_i: $R^N \to R$ by the prescription $\xi_n((t_i)_{i-1}^{\infty}) = t_n$. Then ξ_i is a random variable, $P_{\xi_i} = m_{x_i}$; hence

$$E(\xi_i) = m(x_i), \qquad \sigma^2(\xi_i) = \sigma^2(x_i)$$

and

$$\sum_{n=1}^{\infty} \frac{\sigma^2(\xi_n)}{n^2} < \infty$$

Moreover, by (*)

$$\begin{aligned} &P(\xi_1^{-1}(A_1) \cap \cdots \cap \xi_n^{-1}(A_n)) \\ &\quad = m(h_n(A_1 \times \cdots \times A_n)) \\ &\quad = m(x_1(A_1)) \cdot \ldots \cdot m(x_n(A_n)) = P(\xi_1^{-1}(A_1)) \cdot \ldots \cdot P(\xi_n^{-1}(A_n)) \end{aligned}$$

hence the sequence $(\xi_n)_n$ is independent. Therefore by the classical strong law of large numbers,

$$\left(\frac{1}{n} \sum_{i=1}^{n} (\xi_i - E(\xi_i))\right)_n$$

converges P-a.e. to 0. Define g_n: $R^n \to R$ by the formula

$$g_n(\nu_1, \ldots, \nu_n) = \frac{1}{n} \sum_{i=1}^{n} (\nu_i - E(\xi_i)) = \frac{1}{n} \sum_{i=1}^{n} (\nu_i - m(x_i))$$

Then by the preceding

$$(g_n(\xi_1, \ldots, \xi_n))_n$$

converges P-a.e. to 0. By Riečan (n.d.-d), Corollary 2, then

$$(g_n(x_1, \ldots, x_n))_n$$

converges m-a.e. to 0, too. But by the definition

$$g_n(x_1, \ldots, x_n) = h_n \circ g_n^{-1} = \frac{1}{n} \sum_{i=1}^{n} (x_i - m(x_i))$$

Hence

$$\frac{1}{n} \sum_{i=1}^{n} (x_i - m(x_i))$$

converges m-a.e. to 0.

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